Sequential Quantile Prediction of Time Series.

Joined work with Gérard Biau.

Benoît Patra, benoit.patra@lokad.com

JDS BORDEAUX, MAY 2009.

Introduction.

Time series prediction.

- Time series prediction has a long history (`Yule, 1927`).
- Parametric approaches (Until 70's).
- Recently non parametric approach.

Quantile forecasting.

Given a stochastic process $Y_1, Y_2, \ldots$.

- Usually, estimate the conditional mean of Y_n given $Y_1, \ldots, Y_{n-1}$.
- Here: the conditional τth quantile of Y_n given $Y_1, \ldots, Y_{n-1}$.

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What for?

- Understand conditional distributions.
- $\tau = 0.5$ robust forecasting.
- Build confidence interval.

Applications fields.

- Finance: CVAR. Also biology, medecine, telecoms...
- Here: call volumes (optimize staff in a call center).

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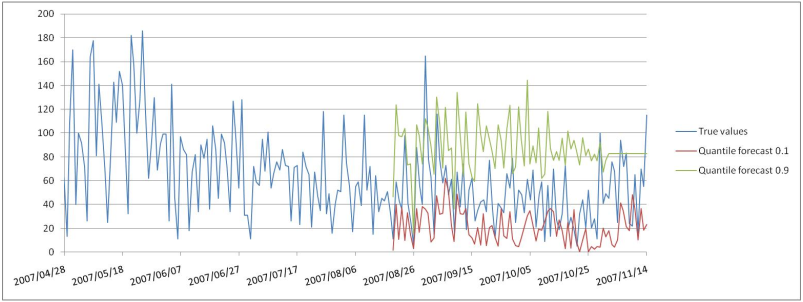

Figure: Quantile forecast with $\tau = 0.1, 0.9$.

Quantile Regression.

Conditional quantiles.

X multivariate random variable, Y real valued random variable,

$$q_\tau(X) \triangleq F^{\leftarrow}_{Y|X}(\tau) = \inf\{t \in \mathbb{R} : F_{Y|X}(t) \geq \tau\}.$$

$F_{Y|X}$ conditional cumulative distribution function.

Proposition (Koenker, 2005)

$$q_\tau(X) \in \operatorname*{argmin}_{q(.)\in\mathbb{R}} \mathbb{E}_{\mathbb{P}_{Y|X}} \left[\rho_\tau\left(Y - q(X)\right)\right].$$

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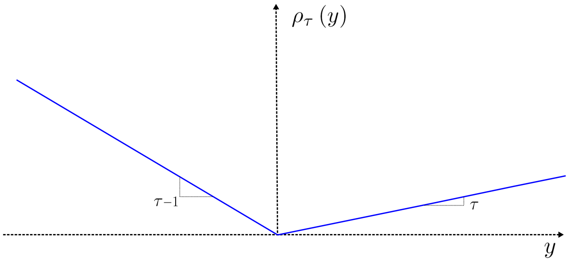

Figure: Pinball function graph.

Non parametric framework.

On line.

- Consider the sequential (= on-line) quantile prediction of time series.
- Including series that do not necessarily satisfy classical statistical assumptions for bounded, mixing or Markovian process.

Goal.

- Show consistency results under a minimum of hypotheses.

Notation.

- $y_1^n = (y_1, \ldots, y_n)$ real sequence.
- $Y_1^n = (Y_1, \ldots, Y_n)$ random variables sequence.

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Framework.

- Here, we observe a string realization y_1^{n-1} of a stationary and ergodic process $\{Y_n\}_{-\infty}^{\infty}\dots$
- $\dots$ and try to estimate $q_\tau(y_1^{n-1}) = F^{\leftarrow}_{Y_n|Y_1^{n-1}=y_1^{n-1}}(\tau)$, the conditional quantile at time n.

Strategy.

Sequence $g = \{g_n\}_{n=1}^{\infty}$ of τth quantile forecasting functions

$$g_n : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}.$$

Quantile estimation at time n is $g_n(y_1^{n-1})$.

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Errors.

Empirical measure criterion.

At time n the cumulative pinball error of the strategy g is

$$L_n(g) = \frac{1}{n}\sum_{t=1}^{n} \rho_\tau \left(y_t - g_t(y_1^{t-1}) \right).$$

A fundamental limit (Algoët, 1994).

For any stationary and ergodic process $\{Y_n\}_{n=-\infty}^{+\infty}$,

$$\liminf_{n\to\infty} L_n(g) \geq L^\star \quad \text{a.s.},$$

where

$$L^\star = \mathbb{E}\left[\min_{q(.)} \mathbb{E}_{\mathbb{P}_{Y_0|Y_{-\infty}^{-1}}}\left[\rho_\tau\left(Y_0 - q(Y_{-\infty}^{-1})\right)\right]\right].$$

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A NN based aggregation scheme.

On line learning.

Scheme inspired from prediction of individual sequences.
Reference : Cesa-Bianchi, N. and Lugosi, G. Prediction, Learning, and Games, Cambridge University Press, New York, 2006.

Previous works.

- Sequential prediction of bounded time series. Györfi, Lugosi, 2001.
- ... portfolio optimization. Györfi, Lugosi, 2006.
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Nearest neighbors strategy.

Elementary predictors.

- Define infinite array of experts $h_n^{(k,\ell)}$: $k, \ell = 1, 2, \dots$

Each expert has a job.

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?

Figure: Work of fundamental expert with $k = 3$ and $\bar{\ell} = 4$.

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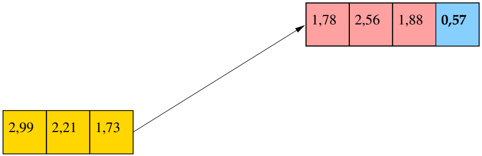

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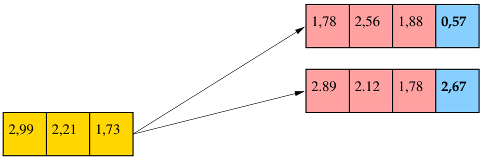

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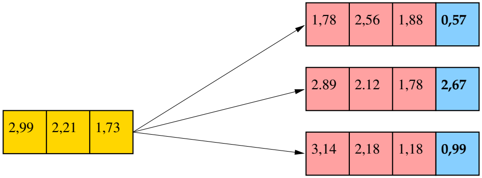

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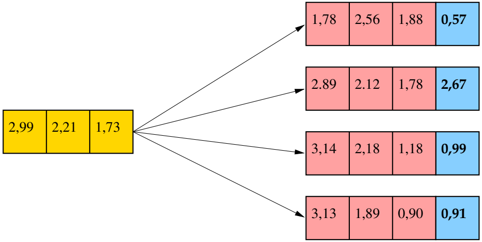

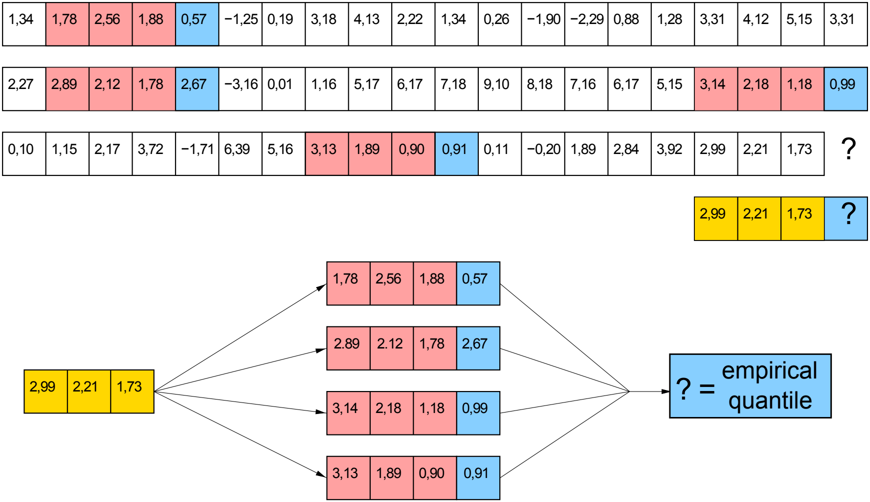
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Prediction and Aggregation.

Prediction of one expert.

$$h_n^{(k,\ell)}(y_1^{n-1}) = \operatorname*{argmin}_{q\in\mathbb{R}} \sum_{\{t\in J_n^{(k,\ell)}\}} \rho_\tau(y_t - q).$$

[Can be easily computed by sorting the sample.]

Aggregated prediction of all experts.

$$g_n(y_1^{n-1}) = \sum_{k,\ell=1}^{\infty} p_{k,\ell,n} h_n^{(k,\ell)}(y_1^{n-1}).$$

Where do the $p_{k,\ell,n}$ come from?

Exponentially weight the experts based on their past performance.

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Aggregation continued. . .

Definitions.

- Let $\{q_{k,\ell}\}$ be a probability distribution over all pairs (k, ℓ) of positive integers such that $q_{k,\ell} > 0$ for all (k, ℓ).
- For $\eta_n > 0$, we define the weights

$$w_{k,\ell,n} = q_{k,\ell} e^{-\eta_n (n-1) L_{n-1}(h_n^{(k,\ell)})}.$$

- We normalize these weights:

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Global prediction.

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Theoretical Results.

Theorem

- *Let $\mathcal{C}$ be the class of all jointly stationary and ergodic processes $\{Y_n)\}_{-\infty}^{\infty}$ such that $\mathbb{E}\{Y_0^2\} < \infty$ and $F_{Y_0|Y_{-\infty}^{-1}}$ is a.s. increasing.*
- *Then the nearest neighbor quantile forecasting strategy is universally consistent with respect to the class $\mathcal{C}$, that is, for all process $Y \in \mathcal{C}$*

$$\lim_{n\to\infty} L_n(g) = L^{\star} \quad \text{almost surely.}$$

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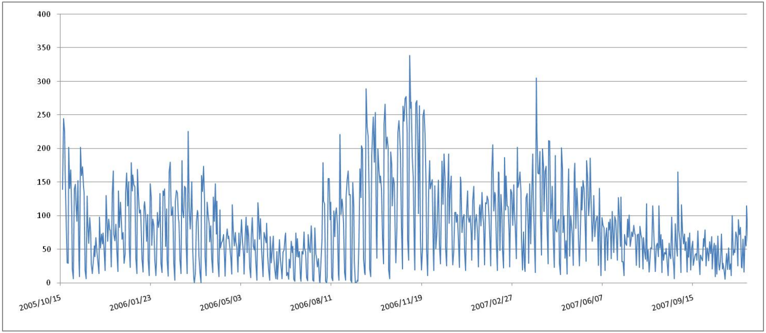
Call center data set.

- Daily call volumes entering a call center.
- Long series between 382 and 826 time values. 21 series.



Future outcome predictions results.

$\tau = 0.5$ median base forecaster : robustness.

Model Name	Avg Abs Error	Avg Sqr Error	Mape (%)
$AR(7)$	65.80	9738	31.6
$QAR(8)_{0.5}$	57.8	9594	24.9
DayOfTheWeekMean	53.95	7099	22.8
HoltWinters	49.84	6025	**21.5**
$\underline{QuantileExpertMixture_{0.5}}$	**48.1**	**5731**	21.6
MeanExpertMixture	52.37	6536	22.3
MA	179	62448	0.52

Figure: Forecasting future outcomes.

Quantile forecasting.

Model Name	PinBall Loss (0.1)	Ramp Loss
$QuantileExpertMixture_{0.1}$	13.71	0.80
$QAR(7)_{0.1}$	**13.22**	**0.88**

Figure: $\tau = 0.1$

Model Name	PinBall Loss (0.9)	Ramp Loss
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$QAR(7)_{0.9}$	19.31	**0.07**

Figure: $\tau = 0.9$

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